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A Novel Construction of the g-Riesz Decomposition in Hilbert Spaces

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Abstract: The concept of the g-frame, a generalized frame in Hilbert spaces, has garnered attention in recent research. While numerous properties of g-frames have been explored, certain aspects remain unexamined, including a novel construction approach for the g-Riesz decomposition in Hilbert spaces. While prior works such as [12] presented the equivalence conditions for g-Riesz decomposition and Khosravi [10] proposed a new construction method for g-frames, neither addressed a new construction method for g-Riesz decomposition. This paper aims to fill this gap by investigating a novel construction method for the specialized g-frame-g-Riesz decomposition. Leveraging operator theory from generalized functional analysis and function space techniques in complex Hilbert spaces, we establish necessary and sufficient conditions for constructing g-Riesz decompositions, an area insufficiently explored by Khosravi and [12]. Furthermore, we introduce two annotations and provide proofs demonstrating that g-Riesz bases are equivalent to Riesz bases, aligning with the findings of W.C. Sun in [6]. This underscores the significance of our research. The newly proposed g-Riesz decomposition not only contributes to mathematical inquiry but also holds promise for various applications, particularly in signal and image processing.

Keywords: g-framework; g-Riesz decomposition; g-Riesz basis; Riesz basis

1. Introduction

The concept of frames in Hilbert spaces was initially introduced by Duffin and Schaeffer [1] in 1952, within the scope of their study on nonconcordant Fourier series. However, it wasn't until 1986, when Daubechies et al. demonstrated that frames could expand functions in $L^2(R)$ into a similar standard orthogonal basis, that they garnered significant attention. Over the years, substantial research outcomes have emerged in the field of frame theory [2-5], with ongoing generalizations of frames. Professor W. C. Sun [6,7] pioneered the definitions of g-framework and g-Riesz basis, leading to several important findings. Additionally, Professor Y. C. Zhu[8,9] introduced the concept of pre-frame operator Q and utilized it to establish g-frames, g-Riesz bases, and related topics.

Khosravi [10] investigates the redundancy of g-frames using g-Riesz decomposition, and explores staggered dyadic g-frames and g-frame perturbations. Casazza [11] provides the definition of Riesz decomposition and derives relevant properties. Furthermore, Khosravi [10] presents a novel construction of g-frames, laying the groundwork for a new approach to the special g-frame-g-Riesz decomposition.

2. Literature Review

This paper delves into a novel construction method for the special g-frame-g-Riesz decomposition in complex Hilbert spaces. To establish a comprehensive understanding, it is imperative to introduce pertinent definitions of linear spaces, complex Hilbert spaces, standard orthogonal bases, frames, Riesz bases, nonredundant frames, g-frames, g-Riesz bases, g-Riesz decompositions, and associated lemmas. For further elucidation, refer to the following references: [1], [4], [7], [8], [12], etc.

In this paper, we use the following notation: let U,V be the sequence of closed subspaces of two complex Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$, paradigm $\|\cdot\|$, and $\{V_i\}_{i\in I}$ for V, where I is a subset of the set of integers Z, and $L(U,V_i)$ denotes the entirety of all bounded linear operators from U to V_i. Define the linear space

$$l^2(\{V_i\}_{i\in I})=\left\{\{f_i\}_{i\in I}: f_i\in V_i,\forall i\in I,\text{且}\sum_{i\in I}\|f_i\|^2<+\infty\right\},$$

Define the inner product on which: the

$$\langle\{f_i\}_{i\in I},\{g_i\}_{i\in I}\rangle=\sum_{i\in I}\langle f_i,g_i\rangle$$

Then $l^2(\{V_i\}_{i\in I})$ is a complex Hilbert space.

Let $\{e_{ij}\}_{j\in J_i}$ be the standard orthogonal basis of V_i, where J_i is a subset of the set of integers, $i\in I$. Let

$\tilde{e}_{ij}=\{\delta_{ik}e_{kj}\}_{k\in I}$, $i\in I$, $j\in J_i$.

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Then it can be verified that $\{\tilde{e}_{ij}\}_{i\in I, j\in J_i}$ is the standard orthogonal base of $l^2(\{V_i\}_{i\in I})$.

Definition 1 The sequence $\{f_i\}_{i=1}^{\infty} \subset U$ is called the frame of U if there exists a positive number A, B such that for any $f \in U$, there are

$$A\|f\|^2 \le \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \le B\|f\|^2,$$

established, A, B are called the lower and upper bounds of the frame, respectively.

Definition 2 The sequence $\{f_i\}_{i=1}^{\infty} \subset U$ is called the Riesz basis of U if $U = \overline{\text{span}}\{f_i\}_{i=1}^{\infty}$, and there exists a positive number A, B , for any finite constant column $\{a_i\}_{i=1}^{n}$, there are

$$A\sum_{i=1}^{n} |a_i|^2 \le \left\|\sum_{i=1}^{n} a_i f_i\right\|^2 \le B\sum_{i=1}^{n} |a_i|^2$$

established, A, B are called the lower and upper bounds of the Riesz basis, respectively.

Definition 3 If $\{f_i\}_{i=1}^{\infty} \subset U$ is called a non-redundant frame of U , it is not a frame of U if any of their elements are removed.

Definition 4 Let $\Lambda_i \in L(U, V_i)$, $(i \in I)$, and the sequence $\{\Lambda_i\}_{i\in I}$ be called the g-frame of U with respect to $\{V_i\}_{i\in I}$ if there exists a positive number $A,\ B$ such that, for any $f \in U$, there are

$$A\|f\|^2 \le \sum_{i\in I} \|\Lambda_i f\|^2 \le B\|f\|^2 \qquad (1\text{-}1)$$

holds, and $A,\ B$ is called the lower and upper bounds of $\{\Lambda_i\}_{i\in I}$, respectively.

If $A = B = \lambda$, then $\{\Lambda_i\}_{i\in I}$ is said to be the g- λ -tight frame of U with respect to $\{V_i\}_{i\in I}$; if $\lambda = 1$, then $\{\Lambda_i\}_{i\in I}$ is said to be the g-Parseval frame of U with respect to $\{V_i\}_{i\in I}$; and if only the inequality on the right-hand side of Eq. (1-1) holds, then $\{\Lambda_i\}_{i\in I}$ is said to be the g-Bessel sequence of U with respect to $\{V_i\}_{i\in I}$ and the g-Bessel sequence bounded by B .

Definition 5 Let $\Lambda_i \in L(U, V_i)$, $(i \in I)$, and the sequence $\{\Lambda_i\}_{i\in I}$ be called the g-Riesz basis of U with respect to $\{V_i\}_{i\in I}$ if the following two conditions are satisfied:

(1) $\{\Lambda_i\}_{i\in I}$ is g-complete, i.e., $\{f \in U : \Lambda_i f = 0, i \in I\} = \{0\}$;

(2) There exists a positive number $A,\ B$ such that for any finite subset $I_1 \subset I$ and any $f_i \in V_i$, $(i \in I_1)$ there are

$$A\sum_{i\in I_1} \|f_i\|^2 \le \|\sum_{i\in I_1} \Lambda_i^* f_i\|^2 \le B\sum_{i\in I_1} \|f_i\|^2$$

holds, and $A,\ B$ are called the lower and upper bounds of the g-Riesz basis, respectively.

Definition 6 Let , $\Lambda_i \in L(U, V_i)$ $(i \in I)$, and $\{\Lambda_i\}_{i\in I}$ be g-Bessel sequences of U with respect to $\{V_i\}_{i\in I}$, and say that $\{\Lambda_i^*(V_i)\}_{i\in I}$ is a g-Riesz decomposition of U if for any $f \in U$, there exists a unique $\{f_i\}_{i\in I} \in l^2(\{V_i\}_{i\in I})$ such that $f = \sum_{i\in I} \Lambda_i^*(f_i)$.

Lemma 1 The [8] sequence $\{\Lambda_i\}_{i\in I}$ is the g-frame of U with respect to $\{V_i\}_{i\in I}$ if and only if the bounded linear operator Q is full, where Q is $Q : l^2(\{V_i\}_{i\in I}) \to U$, $\{f_i\}_{i\in I} \mapsto \sum_{i\in I} \Lambda_i^*(f_i)$ and the g-frames bounded by $\|Q^+\|^{-2}$ and $\|Q\|^2$, and Q^+ is the pseudo-inverse operator of Q .

Lemma 2 [1,7] Let the sequence $\{f_i\}_{i=1}^{\infty} \subset U$, then the following conditions are equivalent.

(1) $\{f_i\}_{i=1}^{\infty}$ is the Riesz basis of U and the Riesz bound is , $A\ \ B$.

(2) $\{f_i\}_{i=1}^{\infty}$ is the frame of U and the Riesz bound is , $A\ \ B$. and $\{f_i\}_{i=1}^{\infty}$ is l^2 linearly independent, i.e. if $\sum_{i=1}^{\infty} a_i f_i = 0,\ \{a_i\}_{i=1}^{\infty} \in l^2$, then $a_i = 0,\ i \in N$.

(3) $\{f_i\}_{i=1}^{\infty}$ is a non-redundant frame of U and the frame boundary is $A,\ B$.

Lemma 3 [4] The sequence $\{f_i\}_{i=1}^{\infty}\subset U$ is a Riesz basis for U and the Riesz bound is $A,\ B$ if and only if Eq. defines the boundedness operator T to be a linear homomorphism and satisfies $A\|a\|^2\le\|Ta\|^2\le B\|a\|^2,\quad a\in l^2$. where T is

$$T:\{a_i\}_{i=1}^{\infty}\to\sum_{i=1}^{\infty}a_if_i \qquad \{a_i\}_{i=1}^{\infty}\in l^2$$

Lemma 4 [12] Let $\Lambda_i\in L(U,V_i),\ \ J_i$ be a subset of the set of integers and $\{e_{ij}\}_{j\in J_i}$ be the standard orthogonal basis of V_i , where $i\in I$. If $\{\ \Lambda_i\ \}_{i\in I}$ is the g-Bessel sequence of U with respect to $\{\ V_i\ \}_{i\in I}$, then the following conditions are equivalent.

(1) $\{\Lambda_i^*(V_i)\}_{i\in I}$ is the g-Riesz decomposition of U ;

(2) $\{\ \Lambda_i\ \}_{i\in I}$ is the g-Riesz base of U on $\{\ V_i\ \}_{i\in I}$.

3 Description of the scope of the study

In this paper, we give a novel construction method of g-Riesz decomposition and get the conclusion that Riesz basis and g-Riesz decomposition are equivalent,g-Riesz basis and Riesz basis are equivalent, which is consistent with the conclusion given by W.C. Sun in the literature [6].

A new construction method for g-Riesz decomposition

Theorem 3.1 Let $\Lambda_i\in L(U,V_i)$, $(i\in I)$, and let $\{W_{ij}\}_{j\in J_i}$ be a sequence of closed subspaces of the Hilbert space K . For each fixed $i\in I$, $\Gamma_{ij}\in L(V_i,W_{ij})$, $j\in J_i$, $\{\Gamma_{ij}^*(W_{ij})\}_{j\in J_i}$ is the g-Riesz decomposition of V_i , then $\{\Lambda_i^*(V_i)\}_{i\in I}$ is the g-Riesz decomposition of U if and only if $\{\Lambda_i^*\Gamma_{ij}^*(W_{ij})\}_{i\in I,j\in J_i}$ is the g-Riesz decomposition of U .

Proof Necessity. Since $\{\Lambda_i^*(V_i)\}_{i\in I}$ is the g-Riesz decomposition of U , then by the definition of g-Riesz decomposition and Lemma 1, $\{\ \Lambda_i\ \}_{i\in I}$ is the g-frame of U with respect to $\{\ V_i\ \}_{i\in I}$, and similarly $\{\Gamma_{ij}\}_{j\in J_i}$ is the g-frame of V_i with respect to $\{W_{ij}\}_{j\in J_i}$. By Theorem 2.2 of Literature [10], $\{\Gamma_{ij}\Lambda_i\}_{i\in I,j\in J_i}$ is the g-frame of U with respect to $\{W_{ij}\}_{i\in I,j\in J_i}$. Then by Lemma 1, for any $f\in U$, there exists $\{h_{ij}\}_{i\in I,j\in J_i}\in l^2(\{W_{ij}\}_{i\in I,j\in J_i})$ such that $f=\sum_{i\in I}\sum_{j\in J_i}\Lambda_i^*\Gamma_{ij}^*h_{ij}$. Thus $\{\Lambda_i^*\Gamma_{ij}^*(W_{ij})\}_{i\in I,j\in J_i}$ satisfies the existence of the decomposition.

On the other hand, for any $f\in U$, if there exist , $\{f_{ij}\}_{i\in I,j\in J_i}$ $\{g_{ij}\}_{i\in I,j\in J_i}\in l^2(\{W_{ij}\}_{i\in I,j\in J_i})$ such that

$$f=\sum_{i\in I}\sum_{j\in J_i}\Lambda_i^*\Gamma_{ij}^*f_{ij}=\sum_{i\in I}\sum_{j\in J_i}\Lambda_i^*\Gamma_{ij}^*g_{ij}$$

imply $f=\sum_{i\in I}\Lambda_i^*\sum_{j\in J_i}\Gamma_{ij}^*f_{ij}=\sum_{i\in I}\Lambda_i^*\sum_{j\in J_i}\Gamma_{ij}^*g_{ij}\Leftrightarrow f=\sum_{i\in I}\Lambda_i^*(\sum_{j\in J_i}\Gamma_{ij}^*f_{ij})=\sum_{i\in I}\Lambda_i^*(\sum_{j\in J_i}\Gamma_{ij}^*g_{ij})$

Since $\{\Lambda_i^*(V_i)\}_{i\in I}$ is the g-Riesz decomposition of U know that $\sum_{j\in J_i}\Gamma_{ij}^*f_{ij}=\sum_{j\in J_i}\Gamma_{ij}^*g_{ij}$, $i\in I$. Also $\{\Gamma_{ij}^*(W_{ij})\}_{j\in J_i}$ is the g-Riesz decomposition of V_i , so we have $f_{ij}=g_{ij}$, $i\in I$, $j\in J_i$. Therefore, $\{\Lambda_i^*\Gamma_{ij}^*(W_{ij})\}_{i\in I,j\in J_i}$ is the g-Riesz decomposition of U .

Sufficiency. If $\{\Lambda_i^*\Gamma_{ij}^*(W_{ij})\}_{i\in I,j\in J_i}$ is the g-Riesz decomposition of U , then by the definition of g-Riesz decomposition and by Lemma 1, $\{\Gamma_{ij}\Lambda_i\}_{i\in I,j\in J_i}$ is the g-frame of U with respect to $\{W_{ij}\}_{i\in I,j\in J_i}$, and similarly $\{\Gamma_{ij}\}_{j\in J_i}$ is the g-frame of V_i with respect to $\{W_{ij}\}_{j\in J_i}$. By Theorem 2.2 of the literature [10], $\{\Lambda_i\}_{i\in I}$ is the g-frame of U with respect to $\{V_i\}_{i\in I}$. By Lemma 1, for any $f\in U$, there exists $\{h_i\}_{i\in I}\in l^2(\{V_i\}_{i\in I})$ such that $f=\sum_{i\in I}\Lambda_i^*h_i$ and hence $\{\Lambda_i^*(V_i)\}_{i\in I}$ satisfy the existence of the decomposition.

On the other hand, for any $f\in U$, if there exist $\{f_i\}_{i\in I}$, $\{g_i\}_{i\in I}\in l^2(\{V_i\}_{i\in I})$ such that

$$f=\sum_{i\in I}\Lambda_i^*f_i=\sum_{i\in I}\Lambda_i^*g_i$$

Since $\{\Gamma_{ij}^*(W_{ij})\}_{j\in J_i}$ is a g-Riesz decomposition of V_i, for the above $\{f_i\}_{i\in I}$, $\{g_i\}_{i\in I}\in l^2(\{V_i\}_{i\in I})$, there exist unique $\{m_{ij}\}_{i\in I,j\in J_i}$, $\{n_{ij}\}_{i\in I,j\in J_i}\in l^2(\{W_{ij}\}_{i\in I,j\in J_i})$ such that

$$f_i=\sum_{j\in J_i}\Gamma_{ij}^*m_{ij} \qquad ,g_i=\sum_{j\in J_i}\Gamma_{ij}^*n_{ij} . \tag{3-1}$$

Thus $f=\sum_{i\in I}\Lambda_i^*\sum_{j\in J_i}\Gamma_{ij}^*m_{ij}=\sum_{i\in I}\Lambda_i^*\sum_{j\in J_i}\Gamma_{ij}^*n_{ij}$, i.e. $f=\sum_{i\in I}\sum_{j\in J_i}\Lambda_i^*\Gamma_{ij}^*m_{ij}=\sum_{i\in I}\sum_{j\in J_i}\Lambda_i^*\Gamma_{ij}^*n_{ij}$.

Also $\{\Lambda_i^*\Gamma_{ij}^*(W_{ij})\}_{i\in I,j\in J_i}$ is the g-Riesz decomposition of U, so we have, $m_{ij}=n_{ij}$, $i\in I$, $j\in J_i$. Combining with equation (3-1), we have, $f_i=g_i$, $i\in I$. Thus, $\{\Lambda_i^*(V_i)\}_{i\in I}$ satisfies the uniqueness of g-Riesz decomposition. Therefore, $\{\Lambda_i^*(V_i)\}_{i\in I}$ is the g-Riesz decomposition of U.

The g-Riesz decomposition is equivalent to the Riesz basis

Remark 3.1 For any $i\in I$, let $\{f_{ij}\}_{j\in J_i}$ be the Riesz basis of V_i and let the Riesz bound be $A,\ B$. Then by Lemma 2, $\{f_{ij}\}_{j\in J_i}$ is the frame of V_i and the frame bound is $A,\ B$. Take the sequence of closed subspaces $\{V_{ij}\}_{j\in J_i}$ of V_i, where $V_{ij}=\overline{\text{span}}\{f_{ij}\}$, $j\in J_i$. Let $\{e_{ij}\}_{j\in J_i}$ be the standard orthogonal basis of V_i, then by the definition of Riesz basis in literature [5], there exists a bounded invertible operator $T_i\in L(V_i)$ such that $f_{ij}=T_ie_{ij}$. Let the bounded linear operator $\Gamma_{ij}:V_i\to V_{ij}$ be as follows $\Gamma_{ij}f_i=\langle f_i,f_{ij}\rangle f_{ij}$, $j\in J_i$, $f_i\in V_i$.

Then for any $f_i\in V_i$, we have

$$A\|T_i^{-1}\|^{-2}\|f_i\|^2\le\sum_{j\in J_i}\|\Gamma_{ij}f_i\|^2=\sum_{j\in J_i}|\langle f_i,f_{ij}\rangle|^2\|f_{ij}\|^2=\sum_{j\in J_i}|\langle f_i,f_{ij}\rangle|^2\|T_ie_{ij}\|^2\le B\|T_i\|^2\|f_i\|^2$$

So there is $\{\Gamma_{ij}\}_{j\in J_i}$ for the g-frame of V_i about $\{V_{ij}\}_{j\in J_i}$ and the frame boundaries are $A\|T_i^{-1}\|^{-2}$, $B\|T_i\|^2$.

The following verifies that $\{f_{ij}\}_{j\in J_i}$ is a g-Riesz decomposition of V_i.

Indeed, for any $f_i\in V_i$, $f_{ij}\in V_{ij}$, $j\in J_i$, there are

$$\langle\Gamma_{ij}^*f_{ij},f_i\rangle=\langle f_{ij},\Gamma_{ij}f_i\rangle=\langle f_{ij},\langle f_i,f_{ij}\rangle f_{ij}\rangle=\langle f_{ij},f_i\rangle\|f_{ij}\|^2=\langle\|f_{ij}\|^2 f_{ij},f_i\rangle$$

So we have $\Gamma_{ij}^*f_{ij}=\|f_{ij}\|^2 f_{ij}$, $f_{ij}\in V_{ij}$, $j\in J_i$. Thus we have $\{\Gamma_{ij}^*(V_{ij})\}_{j\in J_i}=\{f_{ij}\}_{j\in J_i}$.

For any $f_i\in V_i$, since $\{\Gamma_{ij}\}_{j\in J_i}$ is the g-frame of V_i with respect to $\{V_{ij}\}_{j\in J_i}$, and combining this with Lemma 1, we know that there exists $\{g_{ij}\}_{j\in J_i}\in l^2(\{V_{ij}\}_{j\in J_i})$ such that $f_i=\sum_{j\in J_i}\Gamma_{ij}^*g_{ij}$.

If there exist $\{h_{ij}\}_{j\in J_i}\in l^2(\{V_{ij}\}_{j\in J_i})$, $\{k_{ij}\}_{j\in J_i}\in l^2(\{V_{ij}\}_{j\in J_i})$ such that

$$f_i=\sum_{j\in J_i}\Gamma_{ij}^*h_{ij}=\sum_{j\in J_i}\Gamma_{ij}^*k_{ij}\text{, i.e., there is }f_i=\sum_{j\in J_i}\|h_{ij}\|^2 h_{ij}=\sum_{j\in J_i}\|k_{ij}\|^2 k_{ij},$$

Then by Lemma 3, $h_{ij}=k_{ij}$, $j\in J_i$. Therefore $\{\Gamma_{ij}^*(V_{ij})\}_{j\in J_i}=\{f_{ij}\}_{j\in J_i}$ is the g-Riesz decomposition of V_i.

Conversely, if $\{f_{ij}\}_{j\in J_i}$ is the g-Riesz decomposition of V_i, then the above proof process can be reversed to obtain that $\{f_{ij}\}_{j\in J_i}$ is the Riesz basis of V_i.

The g-Riesz basis is equivalent to the Riesz basis

Remark 3.2 Taking $\{\Gamma_{ij}^*(V_{ij})\}_{j\in J_i}=\{e_{ij}\}_{j\in J_i}$ to be the standard orthogonal basis of V_i in Theorem 3.1 (and $\{e_{ij}\}_{j\in J_i}$ to be a special g-Riesz decomposition by Remark 3.1), we have that $\{\Lambda_i^*(V_i)\}_{i\in I}$ is the g-Riesz decomposition of U if and only if $\{\Lambda_i^*\Gamma_{ij}^*(V_{ij})\}_{i\in I,j\in J_i}=\{\Lambda_i^*(e_{ij})\}_{i\in I,j\in J_i}=\{u_{ij}\}_{i\in I,j\in J_i}$ is the g-Riesz decomposition of U. Using Lemma 4 and Remark 3.1, we know that $\{\Lambda_i\}_{i\in I}$ is the g-Riesz basis of U with respect to $\{V_i\}_{i\in I}$ if and only if $\{u_{ij}\}_{i\in I,j\in J_i}$ is the Riesz basis of U. This agrees with the conclusion of Theorem 3.1 of [6] in the literature.

4 Results and Discussion

This paper aims to investigate a novel construction method of g-Riesz decomposition in Hilbert space. We establish the sufficient and necessary conditions for constructing g-Riesz decomposition, a topic previously unexplored by researchers.

Additionally, by providing illustrative examples [6], we arrive at conclusions consistent with those found in the literature [6], underscoring the significance of our research. Nonetheless, our analysis does not delve into the stability of g-Riesz decomposition, an aspect meriting further comprehensive examination.

5. Conclusion

Based on the exploration conducted in this paper, a novel construction method for the g-Riesz decomposition in complex Hilbert spaces has been introduced. Leveraging operator theory and function space techniques, the authors have established necessary and sufficient conditions for constructing g-Riesz decompositions, a topic insufficiently explored in prior research. The investigation has revealed that g-Riesz bases are equivalent to Riesz bases, consistent with the findings of W.C. Sun. The newly proposed g-Riesz decomposition not only contributes to mathematical inquiry but also holds promise for various applications, particularly in signal and image processing.

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REFERENCES

[1] R. J. Duffin and A. C. Schaeffer, "A class of nonharmonic Fourier series," Trans. Amer. Math. Soc., vol. 72, pp. 341-366, 1952.
[2] P. G. Casazza, "The art of frame theory," Taiwanese J. Math., vol. 4, no. 2, pp. 129-201, 2000.
[3] O. Christensen, "Frames, Riesz bases and discrete Gabor/wavelet expansions," Bull. Amer. Math. Soc., vol. 38, no. 3, pp. 273-291, 2001.
[4] O. Christensen, An introduction to frames and Riesz bases. Boston: Birkhäuser, 2003.
[5] D. Li and M. Xue, Base and frame on Banach space. Beijing: Science Press, 2007.
[6] W. Sun, "G-frames and g-Riesz bases," J. Math. Anal. Appl., vol. 322, no. 1, pp. 437-452, 2006.
[7] W.Sun, "Stability of g-frames," J. Math. Anal. Appl., vol. 326, no. 2, pp. 858-868, 2007.
[8] Y. C. Zhu, "Characterizations of g-frames and g-Riesz bases in Hilbert spaces," Acta Math. Sin., vol. 24, no. 10, pp. 1727-1736, 2008.
[9] Y.J.Wang, "Sequences of g-frames and g-Riesz frames in Hilbert spaces," Master's thesis, Fuzhou University, Fujian, 2007.
[10] A. Khosravi and K. Musazadeh, "Fusion frames and g-frames," J. Math. Anal. Appl., vol. 342, no. 2, pp. 1068-1083, 2008.
[11] P. G. Casazza and G. Kutyniok, "Frames of subspaces," Contemp. Math., vol. 345, pp. 87-113, 2004.
[12] Y. Wang and Y. Zhu, "The g-Riesz decomposition in Hilbert space," J. Fuzhou Univ., vol. 38, no. 5, pp. 617-622, 2010.